A CLASS OF SIMPLE WEIGHT VIRASORO MODULES

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Abstract. For a simple module M over the positive part of the Virasoro algebra (actually for any simple module over some finite dimensional solvable Lie algebras $\mathfrak{a}_r$) and any $\alpha \in \mathbb{C}$, a class of weight modules $\mathcal{N}(M, \alpha)$ over the Virasoro algebra are constructed. The necessary and sufficient condition for $\mathcal{N}(M, \alpha)$ to be simple is obtained. We also determine the necessary and sufficient conditions for two such irreducible Virasoro modules to be isomorphic. Many examples for such irreducible Virasoro modules with different features are provided. In particular the irreducible weight Virasoro modules $\Gamma(\alpha_1, \alpha_2, \lambda_1, \lambda_2)$ are defined on the polynomial algebra $\mathbb{C}[x] \otimes \mathbb{C}[t, t^{-1}]$ for any $\alpha_1, \alpha_2, \lambda_1\lambda_2 \in \mathbb{C}$ with λ_1 or λ_2 nonzero. By twisting the weight modules $\mathcal{N}(M, \alpha)$ we also obtain nonweight simple Virasoro modules $\mathcal{N}(M, \beta)$ for any $\beta \in \mathbb{C}[t, t^{-1}]$.

Keywords: Virasoro algebra, weight module

2000 Math. Subj. Class.: 17B10, 17B20, 17B65, 17B66, 17B68

1. Introduction

We denote by $\mathbb{Z}$, $\mathbb{Z}_+$, $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ the sets of all integers, nonnegative integers, positive integers, real numbers and complex numbers, respectively. For a Lie algebra $\mathfrak{a}$ we denote by $U(\mathfrak{a})$ the universal enveloping algebra of $\mathfrak{a}$.

Let $\mathfrak{V}$ denote the complex *Virasoro algebra*, that is the Lie algebra with basis $\{\mathsf{c}, d_i : i \in \mathbb{Z}\}$ and the Lie bracket defined (for $i, j \in \mathbb{Z}$) as follows:

$$[d_i, d_j] = (j - i)d_{i+j} + \delta_{i,-j}\frac{i^3 - i}{12}c; \quad [d_i, c] = 0.$$

The algebra $\mathfrak{V}$ is one of the most important Lie algebras both in mathematics and in mathematical physics, see for example [KR, IK] and references therein. The Virasoro algebra theory has been widely used in many physics areas and other mathematical branches, for example, quantum physics [GO], conformal field theory [FMS], Higher-dimensional WZW models [IKUX, IUK], Kac-Moody algebras [K, MoP], vertex algebras [LL], and so on.

The representation theory on the Virasoro algebra has attracted a lot of attentions from mathematicians and physicists. There are two

Date: Sept.26, 2012.

classical families of simple Harish-Chandra $\mathfrak{V}$-modules: highest weight modules (completely described in [FF]) and the so-called intermediate series modules. In [Mt] it is shown that these two families exhaust all simple weight Harish-Chandra modules. In [MZ1] it is even shown that the above modules exhaust all simple weight modules admitting a nonzero finite dimensional weight space. Very naturally, the next important task is to study simple weight modules with infinite dimensional weight spaces. The first such examples were constructed by taking the tensor product of some highest weight modules and some intermediate series modules in [Zh] in 1997 , and then Conley and Martin gave another class of such examples with four parameters in [CM] in 2001. Since then (for the last decade) there were no other such irreducible Virasoro modules found. The present paper is to construct a huge class of such irreducible Virasoro modules.

At the same time for the last decade, various families of nonweight simple Virasoro modules were studied in [OW1, LGZ, LZ, FJK, Ya, GLZ, MW, OW2]. These (except the Weyl modules in [LZ]) are basically various versions of Whittaker modules constructed using different tricks. In particular, all the above Whittaker modules (but the ones in [MW])and even more were described in a uniform way in [MZ3]. Now we briefly describe the main results in the present paper.

Denote by $\mathfrak{V}_+$ the Lie subalgebra of $\mathfrak{V}$ spanned by all d_i with $i \geqslant 0$. For $r \in \mathbb{Z}_+$, denote by $\mathfrak{V}_+^{(r)}$ the Lie subalgebra of $\mathfrak{V}$ generated by all d_i, $i > r$, and by $\mathfrak{a}_r$ the quotient algebra $\mathfrak{V}_+/\mathfrak{V}_+^{(r)}$. By $\bar{d}_i$ we denote the image of d_i in $\mathfrak{a}_r$. A classification for all irreducible modules over $\mathfrak{a}_1$ was given in [Bl], while a classification for all irreducible modules over $\mathfrak{a}_2$ was recently obtained in [MZ3]. The problem is open for all other Lie algebras $\mathfrak{a}_r$ for $r \geq 2$.

The paper is organized as follows. In Sect.2, for any module M over $\mathfrak{a}_r$ and any $\alpha \in \mathbb{C}$, we define a weight Virasoro module structure on the vector space $\mathcal{N}(M, \alpha) = V \otimes \mathbb{C}[t^{\pm 1}]$. In Sect.3 we prove that the Virasoro module $\mathcal{N}(M, \alpha)$ is simple if $r \geq 1$ and M is an infinite dimensional irreducible $\mathfrak{a}_r$-module. Thus we obtain a huge class of irreducible weight Virasoro modules. We also determine the necessary and sufficient conditions for two such irreducible Virasoro modules to be isomorphic. In Sect.4 we compare these irreducible weight modules $\mathcal{N}(M, \alpha)$ with other known irreducible weight Virasoro modules ([CM, Zh]) to prove that they are new. In Sect.5 we give concrete examples for such irreducible Virasoro modules $\mathcal{N}(M, \alpha)$. In particular the irreducible weight Virasoro modules $\Gamma(\alpha_1, \alpha_2, \lambda_1, \lambda_2)$ are defined on the polynomial algebra $\mathbb{C}[x] \otimes \mathbb{C}[t, t^{-1}]$ for any $\alpha_1, \alpha_2, \lambda_1\lambda_2 \in \mathbb{C}$ with λ_1 or λ_2 nonzero. In Sect.6, by twisting the weight modules $\mathcal{N}(M, \alpha)$ we also obtain nonweight simple Virasoro modules $\mathcal{N}(M, \beta)$ for any

$\beta \in \mathbb{C}[t, t^{-1}]$. These nonweight irreducible Virasoro modules are also new.

2. Constructing new weight Virasoro modules

Let us start with a module M over $\mathfrak{a}_r$ and $\alpha \in \mathbb{C}$. We define a Virasoro module structure on the vector space $\mathcal{L}(M, \alpha) = V \otimes \mathbb{C}[t^{\pm 1}]$ as follows

def

$$d_m(v \otimes t^n) = ((\alpha + n + \sum_{i=0}^{r} (\frac{m^{i+1}}{(i+1)!}\bar{d}_i))v) \otimes t^{n+m}, \tag{2.1}$$

$$c(v \otimes t^n) = 0, \tag{2.2}$$

where $m, n \in \mathbb{Z}, v \in M$. This will be verified after the next example.

We will always write $v(n) = v \otimes t^n$ for short.

Example 1. When $r = 0$, any irreducible module $M = \mathbb{C}v$ over the Lie algebra $\mathfrak{a}_0$ is one dimensional and given by a scalar $b \in \mathbb{C}$ with the action $\bar{d}_0 v = bv$. In this case, $\mathcal{N}(M, \alpha) = \mathbb{C}v \otimes \mathbb{C}[t^{\pm 1}]$ and (2.1) becomes

$$d_m v(n) = (\alpha + n + bm)v(n+m), \tag{2.3}$$

which is exactly the module of intermediate series (see [KR]).

Proposition 1. *The actions (2.1) and (2.2) make $\mathcal{N}(M, \alpha)$ into a module over $\mathfrak{V}$.*

Proof. We need to verify that $[d_m, d_k]v(n) = (d_m d_k - d_k d_m)v(n)$ for all $m, n \in \mathbb{Z}$ and $v \in M$. By (2.1), we have

$$[d_m, d_k]v(n) = (k - m)(\alpha + n + \sum_{i=0}^{r} \frac{(m+k)^{i+1}}{(i+1)!}\bar{d}_i)v(n+m+k).$$

At the same time

$$\begin{aligned}
&(d_m d_k - d_k d_m)v(n) \\
=&\Big(\alpha + n + k + \sum_{i=0}^{r} \frac{m^{i+1}}{(i+1)!}\bar{d}_i\Big)\Big(\alpha + n + \sum_{j=0}^{r} \frac{k^{j+1}}{(j+1)!}\bar{d}_j\Big)v(n+m+k) \\
-&\Big(\alpha + n + m + \sum_{i=0}^{r} \frac{k^{j+1}}{(j+1)!}\bar{d}_j\Big)\Big(\alpha + n + \sum_{i=0}^{r} \frac{m^{i+1}}{(i+1)!}\bar{d}_i\Big)v(n+m+k) \\
=&(k-m)(\alpha+n)v(n+m+k) \\
&+ \Big(k\sum_{i=0}^{r} \frac{k^{i+1}}{(i+1)!} - m\sum_{i=0}^{r} \frac{m^{i+1}}{(i+1)!}\Big)\bar{d}_i v(n+m+k) \\
&+ \sum_{i,j=0}^{r} \frac{m^{i+1}k^{j+1}}{(i+1)!(j+1)!}(j-i)\bar{d}_{i+j}v(n+m+k)
\end{aligned}$$

$$
\begin{aligned}
=&(k-m)(\alpha+n)v(n+m+k)\\
&+\Big(k\sum_{i=0}^{r}\frac{k^{i+1}}{(i+1)!}-m\sum_{i=0}^{r}\frac{m^{i+1}}{(i+1)!}\Big)\bar{d}_i v(n+m+k)\\
&+\left(k\sum_{i,j=0}^{r}\frac{m^{i+1}k^j}{(i+1)!j!}\bar{d}_{i+j}-m\sum_{i,j=0}^{r}\frac{m^i k^{j+1}}{i!(j+1)!}\bar{d}_{i+j}\right)v(n+m+k)\\
=&(k-m)(\alpha+n)v(n+m+k)\\
&+\Big(k\sum_{i=0}^{r}\frac{k^{i+1}}{(i+1)!}-m\sum_{i=0}^{r}\frac{m^{i+1}}{(i+1)!}\Big)\bar{d}_i v(n+m+k)\\
&+k\sum_{i=0}^{r}\sum_{j=0}^{i}\frac{m^{i+1-j}k^j}{(i+1-j)!j!}\bar{d}_i v(n+m+k)\\
&-m\sum_{i=0}^{r}\sum_{j=0}^{i}\frac{k^{i+1-j}m^j}{(i+1-j)!j!}\bar{d}_i v(n+m+k)\\
=&\ (k-m)(\alpha+n)v(n+m+k)\\
&+k\sum_{i=0}^{r}\sum_{j=0}^{i+1}\frac{m^{i+1-j}k^j}{(i+1-j)!j!}\bar{d}_i v(n+m+k)\\
&-m\sum_{i=0}^{r}\sum_{j=0}^{i+1}\frac{k^{i+1-j}m^j}{(i+1-j)!j!}\bar{d}_i v(n+m+k)\\
=&\ (k-m)\Big(\alpha+n+\sum_{i=0}^{r}\frac{(m+k)^{i+1}}{(i+1)!}\bar{d}_i\Big)v(n+m+k).
\end{aligned}
$$

Then $[d_m, d_k]v(n) = (d_m d_k - d_k d_m)v(n)$. Consequently, $\mathcal{L}(M,\alpha)$ is a module over $\mathfrak{V}$. $\square$

From (2.1), if M is infinite dimensional, then $\mathcal{N}(M,\alpha) = \oplus_{n\in\mathbb{Z}}\mathcal{N}_{\alpha+n}$ is a weight Virasoro module with infinite dimensional weight spaces $\mathcal{N}_{\alpha+n} = M\otimes t^n$ where

$$\mathcal{N}_{\alpha+n} = \{v\in\mathcal{N}(M,\alpha) \mid d_0 v = (\alpha+n)v\}.$$

3. Simplicity and the isomorphism classes of weight Virasoro modules $\mathcal{N}(M,\alpha)$

In this section we will determine the simplicity and the isomorphism classes of $\mathcal{N}(M,\alpha)$.

trivial **Lemma 2.** *Let M be a simple module over $\mathfrak{a}_r$. Then either $\bar{d}_r M = 0$ or the action of $\bar{d}_r$ on M is bijective.*

Proof. It is straightforward to check that $\bar{d}_r M$ and $\mathrm{ann}_M(\bar{d}_r) = \{v\in M|\bar{d}_r v = 0\}$ are submodules of V. Then the lemma follows from the simplicity of M. $\square$

For any $l, m \in \mathbb{Z}$, by induction on $s \in \mathbb{Z}_+$ we define the following important operators

$$\omega_{l,m}^{(0)} = d_{l-m} d_m \in U(\mathrm{Vir}),$$
$$\omega_{l,m}^{(1)} = \omega_{l,m+1}^{(0)} - \omega_{l,m}^{(0)}, \text{ and}$$
$$\omega_{l,m}^{(s)} = \omega_{l,m+1}^{(s-1)} - \omega_{l,m}^{(s-1)}.$$

Thus

$$\omega_{l,m}^{(s)} = \sum_{i=0}^{s} \binom{s}{i} (-1)^{s-i} d_{l-m-i} d_{m+i} \in U(\mathrm{Vir}). \tag{3.1}$$

key-compute **Lemma 3.** *Let $r \geq 0$, M be a module over $\mathfrak{a}_r$. Then for any $m, l \in \mathbb{Z}$, $s \in \mathbb{Z}_+$ we have*

$$\omega_{l,m}^{(2r+2)} v(n) = (2r+2)!(-1)^{r+1} \bar{d}_r^2 v(n+l), \forall v \in M, n \in \mathbb{Z}, \text{ and}$$
$$\omega_{l,m}^{(s)} \mathcal{N}(M, \alpha) = 0, \forall s > 2r+2.$$

Proof. Let us fix $l, n \in \mathbb{Z}$ and $v \in M$. We have

$$\omega_{l,m}^{(0)} v(n) = d_{l-m} d_m v(n)$$
$$= \Big(\alpha + n + m + \sum_{i=0}^{r} \frac{(l-m)^{j+1}}{(j+1)!} \bar{d}_j\Big)\Big(\alpha + n + \sum_{i=0}^{r} \frac{m^{i+1}}{(i+1)!} \bar{d}_i\Big) v(n+l).$$

Note that the righthand of the equation is a polynomial of the variable m and has degree $2r+2$, which can be written as

$$(-1)^{r+1} m^{2r+2} \bar{d}_r^2 v(n+l) + \Big(\sum_{j=0}^{2r+1} m^j v_{j,0}(n+l)\Big),$$

where $v_{j,0} \in U(\mathfrak{a}_r)v$ is independent of m. By induction on s we can easily see that

$$\omega_{l,m}^{(s)} v(n) = (-1)^{r+1}(2r+2)(2r+1)...(2r+3-s)) m^{2r+2-s} \bar{d}_r^2 v(n+l)$$
$$+\Big(\sum_{j=0}^{2r+1-s} m^j v_{j,s}(n+l)\Big),$$

where $v_{j,s} \in U(\mathfrak{a}_r)v$ is independent of m. Now the lemma follows from this formula. □

Let M be a module over $\mathfrak{a}_r$ such that $\mathcal{N}(M, \alpha)$ is simple. From (2.1), we know that M is a simple module over $\mathfrak{a}_r$. From Example 1 and Lemma 2, we may further assume that $r \geq 1$ and the action of $\bar{d}_r$ on M is bijective.

Now we can prove our main result in this section.

Theorem 4. *Let $r \geq 1$, M be a simple module over $\mathfrak{a}_r$ such that the action of $\bar{d}_r$ on M is injective. Then for any $\alpha \in \mathbb{C}$, the weight Virasoro module $\mathcal{N}(M, \alpha)$ is simple.*

Proof. Suppose W is a nonzero Virasoro submodule of $\mathcal{L}(M,\alpha)$. It suffices to show that $W=\mathcal{N}(M,\alpha)$. Since $\mathcal{N}(M,\alpha)=\oplus_{n\in\mathbb{Z}}(M\otimes t^n)$ is a weight Virasoro module, then $W=\oplus_{n\in\mathbb{Z}}W_n\otimes t^n$ where W_n are subspaces of M. Let $W^{(0)}=\cap_{n\in\mathbb{Z}}W_n$. We will prove that $W^{(0)}$ is a nonzero $\mathfrak{a}_r$-submodule.

Claim 1: $W^{(0)}$ is nonzero.

Since $W\neq 0$, we may assume that $W_n\neq 0$ for some $n\in\mathbb{Z}$. Take a nonzero $v\in W_n$. From Lemma 3, we see that

$$\omega_{l,m}^{(2r+2)}v(n)=(2r+2)!(-1)^{r+1}\bar{d}_r^2v(n+l)\in W,\forall l\in\mathbb{Z}.$$

Hence $\bar{d}_r^2v\in W^{(0)}$, which is nonzero since the action of $\bar{d}_r$ on M is injective. Claim 1 follows.

Claim 2: $W^{(0)}$ is an $\mathfrak{a}_r$-submodule.

For any $v\in W^{(0)}$, we have $v(k-m)\in W_{k-m}$ for any $k,m\in\mathbb{Z}$. We know that

$$d_mv(k-m)=\Big(a+k-m+\sum_{i=0}^{r}\frac{m^{i+1}}{(i+1)!}\bar{d}_i\Big)v(k)\in W_k\otimes t^k,$$

yielding

$$\sum_{i=0}^{r}\frac{m^{i+1}}{(i+1)!}\bar{d}_iv\in W_k,\forall\ m\in\mathbb{Z}.$$

Hence $\bar{d}_iv\in W_k$ for all $k\in\mathbb{Z}$ and $i=0,1,2,...,r$. Therefore $W^{(0)}$ is an $\mathfrak{a}_r$-submodule.

Since M is a simple module over $\mathfrak{a}_r$, we see that $W^{(0)}=M$, i.e., $W_n=M$ for all $n\in\mathbb{Z}$. Consequently, $W=\mathcal{N}(M,\alpha)$ and $\mathcal{N}(M,\alpha)$ is a simple Virasoro module □

Theorem 5. *Let $\alpha,\alpha'\in\mathbb{C}$, $r,r'\geq 1$, M,M' be simple modules over $\mathfrak{a}_r$, $\mathfrak{a}_{r'}$ respectively such that the actions of $\bar{d}_r,\bar{d}_{r'}$ on M,M' are injective respectively. Then the Virasoro modules $\mathcal{N}(M,\alpha)$ and $\mathcal{N}(M',\alpha')$ are isomorphic if and only if $\alpha-\alpha'\in\mathbb{Z}$, $r=r'$ and $M\cong M'$ as $\mathfrak{a}_r$ modules.*

Proof. The sufficiency of the condition is clear. Now suppose that $\varphi:\mathcal{N}(M,\alpha)\to\mathcal{N}(M',\alpha')$ is a Virasoro module isomorphism. Comparing the set of weights, we have $\alpha+\mathbb{Z}=\alpha'+\mathbb{Z}$, that is, $\alpha-\alpha'\in\mathbb{Z}$. Say $\alpha=\alpha'+n_0$. Denote

$$\varphi(v(0))=\psi(v)(n_0),\ \forall\ v\in M,$$

where ψ is a bijective linear map from M to M'. Then from $\varphi(\omega_{l,m}^{(s)}v(0))=\omega_{l,m}^{(s)}\varphi(v(0))$ and Lemma 3, we have $r=r'$ and

$$\varphi(\bar{d}_r^2v(l))=\bar{d}_r^2\psi(v)(n_0+l),\forall l\in\mathbb{Z},v\in M.$$

Recall from Lemma 2 that $\bar{d}_r$ is an isomorphism of M. So

$$\varphi(v(l)) = \bar{d}_r^2\psi(\bar{d}_r^{-2}v)(n_0+l), \forall\ n \in \mathbb{Z}, v \in M,$$

where $\bar{d}_r^{-1}$ is the inverse mapping of $\bar{d}_r$.

By taking $l=0$, we get $\bar{d}_r^2\psi(\bar{d}_r^{-2}v) = \psi(v)$ and

$$\varphi(v(l)) = \psi(v)(l+n_0), \forall l \in \mathbb{Z}, v \in M.$$

From $\varphi(d_m v(-m)) = d_m\varphi(v(-m))$, we get

$$\psi(\Big(\alpha - m + \sum_{i=0}^{r}\frac{m^{i+1}}{(i+1)!}\bar{d}_i\Big)v)(n_0)$$

$$= \Big(\alpha - m + \sum_{i=0}^{r}\frac{m^{i+1}}{(i+1)!}\bar{d}_i\Big)\psi(v)(n_0),$$

yielding that

$$\sum_{i=0}^{r}\frac{m^{i+1}}{(i+1)!}(\psi(\bar{d}_i v) - \bar{d}_i\psi(v))) = 0.$$

Since m is arbitrary, we have $\psi(\bar{d}_i v) = \bar{d}_i\psi(v)$ for $i = 0, 1, \ldots, r$. Thus we have proved that ψ is an $\mathfrak{a}_r$ module isomorphism, which completes the proof. □

4. Simple Virasoro modules $\mathcal{N}(M,\alpha)$ are new

Known irreducible weight Virasoro modules with infinite dimensional weight spaces are the ones obtained by some tensor product of an irreducible highest (or lowest) weight module and an irreducible Virasoro module of the intermediate series, see [Zh], and the ones defined in [CM]. Let us first recall these modules.

Let $U := U(\mathfrak{V})$ be the universal enveloping algebra of the Virasoro algebra $\mathfrak{V}$. For any $\dot{c}, h \in \mathbb{C}$, let $I(\dot{c},h)$ be the left ideal of U generated by the set

$$\big\{d_i \mid i>0\big\}\bigcup\big\{d_0 - h\cdot 1, c - \dot{c}\cdot 1\big\}.$$

The Verma module with highest weight $(\dot{c},h)$ for $\mathfrak{V}$ is defined as the quotient $\bar{V}(\dot{c},h) := U/I(\dot{c},h)$. It is a highest weight module of $\mathfrak{V}$ and has a basis consisting of all vectors of the form

$$d_{-i_1}d_{-i_2}\cdots d_{-i_k}v_h;\quad k \in \mathbb{N}\cup\{0\}, i_j \in \mathbb{N}, i_k \geqslant \cdots \geqslant i_2 \geqslant i_1 > 0.$$

Then we have the *irreducible highest weigh module* $V(\dot{c},h) = \bar{V}(\dot{c},h)/J$ where J is the maximal proper submodule of $\bar{V}(\dot{c},h)$.

For $a, b \in \mathbb{C}$, the $\mathfrak{V}$-modules $V_{a,b}$ has basis $\{v_{a+i} | i \in \mathbb{Z}\}$ with trivial central actions and

$$d_i v_{a+k} = (a+k+ib)v_{a+k+i}.$$

It is known that $V_{a,b} \cong V_{a+1,b}$, for all $a, b \in \mathbb{C}$ and that $V_{a,0} \cong V_{a,1}$ if $a \notin \mathbb{Z}$. It is also clear that $V_{0,0}$ has $\mathbb{C}v_0$ as a submodule, we denote the quotient module $V_{0,0}/\mathbb{C}v_0$ by $V'_{0,0}$. Dually, $V_{0,1}$ has $\mathbb{C}v_0$ as a quotient module, and its corresponding submodule $\oplus_{i\neq 0}\mathbb{C}v_i$ is isomorphic to $V'_{0,0}$. For convenience we simply write $V'_{a,b} = V_{a,b}$ when $V_{a,b}$ is irreducible.

Let us recall a result from [Zh].

Theorem 6. *If $\bar{V}(\dot{c}, h)$ is reducible, and if a is transcendental over $\mathbb{Q}(\dot{c}, h, b)$ or a is algebraic over $\mathbb{Q}(\dot{c}, h, b)$ with sufficiently large degree, then $V(\dot{c}, h) \otimes V'_{a,b}$ is irreducible.*

Now let us recall the weight modules from Lemma 2.1 in [CM]. For any $h, b, \gamma, p \in \mathbb{C}$, the Virasoro module $E_h(b, \gamma, p)$ has a basis $\{T_i^k | i \in \mathbb{Z}, k \in \mathbb{Z}_+\}$ with the action given by

$$\begin{aligned} d_n T_i^k =& T_{i+n}^{k+1}(1-e^{nh}) + T_{i+n}^k[b+i+n(p+\gamma-(\gamma+k)e^{nh})] \\ & - e^{nh}\sum_{j=0}^{k-1} T_{i+n}^j n^{k-j+1}\left[\begin{pmatrix} k \\ j-1 \end{pmatrix} + \gamma\begin{pmatrix} k \\ j \end{pmatrix}\right], \\ cE_h(b,&\gamma,p) = 0. \end{aligned}$$

Some sufficient conditions for $E_h(b, \gamma, p)$ to be simple were given in [CM]. Now we can compare the modules.

Theorem 7. *Let $r \geq 1$, $\alpha \in \mathbb{C}$, and M be a simple module over $\mathfrak{a}_r$ such that the action of $\bar{d}_r$ on M is injective. Then the Virasoro module $\mathcal{N}(M, \alpha)$ is not isomorphic either to $V(\dot{c}, h) \otimes V'_{a,b}$ for any $a, b, \dot{c}, h \in \mathbb{C}$ or to $E_h(b, \gamma, p)$ for any $b, h, \gamma, p \in \mathbb{C}$.*

Proof. Let v_1 be the highest weight vector of $V(\dot{c}, h)$, v_2 be a nonzero weight vector of $V'_{a,b}$. By Example 1, $V_{a,b} \cong \mathcal{N}(V, a)$, where $V = \mathbb{C}v$ is a one dimensional over the Lie algebra $\mathfrak{a}_0$ and is given by a scalar $b \in \mathbb{C}$ with the action $\bar{d}_0 v = bv$.

From the fact that v_1 is a highest weight vector and Lemma 3, we have

$$\omega_{l,m}^{(3)}(v_1 \otimes v_2) = v_1 \otimes \omega_{l,m}^{(3)} v_2 = 0, \forall\ m > 0, l > m+3.$$

Hence

$$\omega_{l,m}^{(2r+2)}(v_1 \otimes v_2) = v_1 \otimes \omega_{l,m}^{(2r+2)} v_2 = 0, \forall\ m > 0, l > m+2r+3.$$

However by Lemma 3, the action of $\omega_{m,l}^{(2r+2)}$ is injective on $\mathcal{N}(M, \alpha)$. So we have proved $\mathcal{N}(M, \alpha)$ is not isomorphic to any Virasoro module $V(\dot{c}, h) \otimes V'_{a,b}$.

Next suppose that $\mathcal{N}(M, \alpha) \cong E_h(b, \gamma, p)$ for some h, b, γ, p. Note that $E_h(b, \gamma, p)$ is not simple if $e^h = 1$ (See Page 160 in [CM]). So we have $e^h \neq 1$. Denote

$$E_{\leq k} = \operatorname{span}\{T_i^j \in E_h(b, \gamma, p) \mid j \leq k, i \in \mathbb{Z}\}, \forall\ k \in \mathbb{Z}_+.$$

From

$$\begin{aligned}
&\sum_{i=0}^{s}\binom{s}{i}(-1)^{s-i}(1-e^{(-m-i)h})(1-e^{(m+i)h})\\
=&\sum_{i=0}^{s}\binom{s}{i}(-1)^{s-i}(2-e^{(-m-i)h}-e^{(m+i)h})\\
=&-e^{-mh}\sum_{i=0}^{s}\binom{s}{i}(-1)^{s-i}e^{-ih}-e^{mh}\sum_{i=0}^{s}\binom{s}{i}(-1)^{s-i}e^{ih}\\
=&-\left(e^{mh}(e^{h}-1)^{s}+e^{-mh}(e^{-h}-1)^{s}\right)
\end{aligned}$$

we have

$$\omega_{0,m}^{(s)}T_i^k\in-(e^{mh}(e^h-1)^s+e^{-mh}(e^{-h}-1)^s)T_i^{k+2}+E_{\leq k+1},$$

for all $k,s\in\mathbb{N},i\in\mathbb{Z}$.

From Lemma 3, we see that $e^{mh}(e^h-1)^s+e^{-mh}(e^{-h}-1)^s=0$ for all $s>2r+2,m\in\mathbb{Z}$, which implies that $e^h=1$, a contradiction. So we have proved the theorem. □

5. Examples

From Theorem 4 we know that, to obtain new irreducible weight Virasoro modules $\mathcal{N}(M,\alpha)$, it is enough to construct infinite dimensional irreducible modules M over $\mathfrak{a}_r$ for $r>0$ such that the action of $\bar{d}_r$ on M is injective. These modules M were considered in [MZ3]. Let us recall some of such irreducible $\mathfrak{a}_r$-modules.

Example 2. For any $\alpha_1,\alpha_2,\lambda_1,\lambda_2\in\mathbb{C}$ with λ_1 or λ_2 nonzero, we have the simple $\mathfrak{a}_2$ module $M=\mathbb{C}[x]$ with the action

$$\bar{d}_0f(x)=(x+\alpha_1)f(x),$$
$$\bar{d}_if(x)=\lambda_if(x-i),i=1,2,\forall f(x)\in\mathbb{C}[x].$$

Then we have the induced new irreducible weight Virasoro module $\Gamma(\alpha_1,\alpha_2,\lambda_1,\lambda_2)=\mathbb{C}[x,t,t^{-1}]$ with the action

$$c\cdot x^it^n=0$$

$$d_m\cdot x^it^n=\Big(\big(\alpha_2+n+m(x+\alpha_1)\big)x^i+\frac{m^2}{2}\lambda_1(x-1)^i+\frac{m^3}{6}\lambda_2(x-2)^i\Big)t^{m+n},$$

for all $m,n\in\mathbb{Z},i\in\mathbb{Z}_+$. □

Example 3. Fix $r\in\mathbb{N}$ with $r>2$. Choose a pair (S,λ), where $S\subset\{1,2,\ldots,r\}$ and $\lambda=(\lambda_i)_{i\in S}\in\mathbb{C}^{|S|}$, such that the following conditions are satisfied:

cond1 (I) $r\in S$ and $\lambda_r\neq0$;

cond2 (II) for all $i,j\in S$ with $i\neq j$, $i+j\in S$ implies $\lambda_{i+j}=0$;

cond3 (III) for all $j\in\{1,2,\ldots,r\}\setminus S$, we have $r-j\in S$.

One example is $S = \{\lceil \frac{k}{2} \rceil, \lceil \frac{k}{2} \rceil + 1, \ldots, r\}$ and any λ with $\lambda_r \neq 0$. Another example is $S = \{2, 4, 5\}$ for $r = 5$ and any λ with $\lambda_5 \neq 0$. Our final example in this case is $S = \{3, 4, 6, 7, 8\}$ for $r = 8$ and any λ with $\lambda_8 \neq 0$ and $\lambda_7 = 0$ (note that here we have $3 + 4 \in S$).

Denote by Q_λ the $\mathfrak{a}_r$-module $U(\mathfrak{a}_r)/I$, where I is the left ideal generated by $d_i - \lambda_i$, $i \in S$. It was proved in [MZ3] the $\mathfrak{a}_r$-module Q_λ is simple. Then we have irreducible Virasoro modules $\mathcal{N}(Q_\lambda, \alpha)$ for any $\alpha \in \mathbb{C}$. □

Example 4. All irreducible modules over $\mathfrak{a}_1$ were classified in [Bl]. For example, a class of such irreducible $\mathfrak{a}_1$-modules are of the form $M_n = U(\mathfrak{a}_1)/I$ where I is the left ideal generated by $\bar{d}_0^n \bar{d}_1 - 1$ for any fixed $n \in \mathbb{N}$ (see [AP] for details). Then we have irreducible Virasoro modules $\mathcal{N}(M_n, \alpha)$ for any $\alpha \in \mathbb{C}$. □

Example 5. All irreducible modules over $\mathfrak{a}_2$ were classified in [MZ3]. Let $\psi : \mathfrak{a}_2 \to \mathfrak{a}_1$ be the unique Lie algebra epimorphism which sends

$$\bar{d}_0 \to \frac{1}{2}\bar{d}_0, \quad \bar{d}_1 \to 0, \quad \bar{d}_2 \to \bar{d}_1.$$

For any $\lambda \in \mathbb{C}$, mapping

$$\bar{d}_0 \to \bar{d}_0, \quad \bar{d}_1 \to \bar{d}_1, \quad \bar{d}_2 \to \lambda \bar{d}_1^2$$

extends to an epimorphism $\varphi_\lambda : U(\mathfrak{a}_2) \to U(\mathfrak{a}_1)$.

For any simple $\mathfrak{a}_1$-module M, we have two different ways to make M into an irreducible $\mathfrak{a}_2$-modules: $xv = \psi(x)v$ for all $x \in \mathfrak{a}_2$ and $v \in M$; or $xv = \varphi_\lambda(x)v$ for all $x \in \mathfrak{a}_2$ and $v \in M$. These exhaust all irreducible $\mathfrak{a}_2$-modules. Then we have irreducible Virasoro modules $\mathcal{N}(M, \alpha)$ for any $\alpha \in \mathbb{C}$. Modules in Example 2 are special cases. □

6. Deformation of the modules $\mathcal{N}(M, 0)$

In this section we will use the irreducible Virasoro modules $\mathcal{N}(M, 0)$ and "the twisting technique" to construct new irreducible non-weight modules over the Virasoro algebra $\mathfrak{V}$ with trivial action of the center. Let us first recall the definition of the twisted Heisenberg-Virasoro algebra.

The twisted Heisenberg-Virasoro algebra $\mathcal{L}$ is the universal central extension of the Lie algebra $\{f(t)\frac{d}{dt} + g(t) | f, g \in \mathbb{C}[t, t^{-1}]\}$ of differential operators of order at most one on the Laurent polynomial algebra $\mathbb{C}[t, t^{-1}]$. More precisely, the twisted Heisenbeg-Virasoro algebra $\mathcal{L}$ is a Lie algebra over $\mathbb{C}$ with the basis

$$\{d_n, t^n, z_1, z_2, z_3 \mid n \in \mathbb{Z}\}$$

and subject to the Lie bracket

$$[d_n, d_m] = (m-n)d_{n+m} + \delta_{n,-m}\frac{n^3-n}{12}z_1, \tag{6.1}$$

$$[d_n, t^m] = mt^{m+n} + \delta_{n,-m}(n^2+n)z_2, \tag{6.2}$$

$$[t^n, t^m] = n\delta_{n,-m}z_3, \tag{6.3}$$

$$[\mathcal{L}, z_1] = [\mathcal{L}, z_2] = [\mathcal{L}, z_3] = 0. \tag{6.4}$$

The Lie algebra $\mathcal{L}$ has a Virasoro subalgebra $\mathfrak{V}$ with basis $\{d_i, z_1 | i \in \mathbb{Z}\}$, and a Heisenberg subalgebra $\tilde{\mathcal{H}}$ with basis $\{t^i, z_3 \mid i \in \mathbb{Z}\}$.

Let us start with a Virasoro module $\mathcal{N}(M, 0)$ defined in (2.1) and (2.2) with $\alpha = 0$. On $\mathcal{N}(M, 0)$ We can extend the Virasoro structure to a module structure over $\mathcal{L}$ as follows

$$t^k(v \otimes t^n) = v \otimes t^{n+k}, \forall\ k, n \in \mathbb{Z}, v \in M,$$

$$z_1\mathcal{N}(M, 0) = 0, z_2\mathcal{N}(M, 0) = 0, z_3\mathcal{N}(M, 0) = 0.$$

For any $\beta \in \mathbb{C}[t, t^{-1}]$, by using the twisting technique (see (2.3) in [LGZ]) we define the new action of $\mathfrak{V}$ on $\mathcal{N}(M, 0)$ by

$$d_n \circ v = (d_n + \beta t^n)v, \forall\ n \in \mathbb{Z}, v \in \mathcal{N}(M, 0), \tag{6.5}$$

$$c \circ \mathcal{N} = 0. \tag{6.6}$$

We denote by $\mathcal{N}(M, \beta)$ the resulting module over $\mathfrak{V}$.

If $\beta \in \mathbb{C}$, we can easily see that $\mathcal{N}(M, \beta)$ is exactly the modules define in Sect.2. If M is one dimensional, it is not hard to see that $\mathcal{N}(M, \beta) = \mathbb{C}[t, t^{-1}]$ with actions

$$d_m t^n = (\beta + n + bm)t^{n+m}, \forall m, n \in \mathbb{Z}$$

for some $b \in \mathbb{C}$. These are exactly the modules studied in [LGZ]. Recall that this module is reducible if and only if $b = 0$ and $\alpha \in \mathbb{Z}$, or $b = 1$ and $\alpha \in \mathbb{Z} \cup (\mathbb{C}[t^{\pm}] \setminus \mathbb{C})$, see [LGZ] for the details.

Now we can prove the irreducibility of the modules $\mathcal{N}(M, \beta)$.

Theorem 8. *Let $r \geq 1$, M be a simple module over $\mathfrak{a}_r$ such that the action of $\bar{d}_r$ on M is injective. Then for any $\beta \in \mathbb{C}[t, t^{-1}] \setminus \mathbb{C}$, the Virasoro module $\mathcal{N}(M, \beta)$ is simple.*

Proof. Suppose that $W \neq 0$ is a Virasoro submodule of $\mathcal{N}(M,\beta)$. It suffices to show that $W = \mathcal{N}(M,\beta)$. Let us fix a nonzero $w = \sum_{i\in\mathbb{Z}} v_i \otimes t^i \in W$, where the sum is finite. We will use the element $\omega^{(0)}_{l,m}$ defined in (3.1). For any $l, m \in \mathbb{Z}$, the vector $\omega^{(0)}_{l,m} \circ w$ can be written as

$$\begin{aligned}
\omega^{(0)}_{l,m} \circ w &= d_{l-m} \circ d_m \circ w \\
&= (d_{l-m} + \beta t^{l-m})(d_m + \beta t^m) w \\
&= \omega^{(0)}_{l,m} w + d_{l-m}\beta t^m w + \beta t^{l-m} d_m w + \beta^2 t^l w \\
&= \omega^{(0)}_{l,m} w + \sum_{i=0}^{r+2} m^j w_{j,0},
\end{aligned}$$

where $w_{j,0} \in \mathcal{N}(M,0)$ is independent of m. By induction on s we can prove that

$$\omega^{(s)}_{l,m} \circ w = \omega^{(s)}_{l,m} w + \sum_{i=0}^{r+2-s} m^j w_{j,s}$$

where $w_{j,s} \in \mathcal{N}(M,0)$ is independent of m. Using Lemma 3 we see that

$$\omega^{(2r+2)}_{l,m} \circ w = \omega^{(2r+2)}_{l,m} w = (2r+2)!(-1)^{r+1} \sum_i \bar{d}_r^2 v_i \otimes t^{i+l} \in W, \tag{6.7}$$

for all $l, m \in \mathbb{Z}$. In particular, $W' = \{w \in W | t^i w \in W, \forall i \in \mathbb{Z}\}$ is nonzero. Now for any $v \in W'$, we have $d_i v = d_i \circ v - t^i \beta v \in W$, and

$$t^j(d_i v) = -j t^{i+j} v + d_i \circ (t^j v) - t^{i+j}\beta v \in W, \quad \forall\, i, j \in \mathbb{Z}.$$

So $d_i v \in W'$ for all $v \in W'$ and $i \in \mathbb{Z}$. Hence W' is a nonzero submodule of the simple Virasoro module $\mathcal{N}(M,0)$, which has to be $\mathcal{N}(M,0) = \mathcal{N}(M,\beta)$. Therefore $W' = \mathcal{N}(M,\beta)$, and consequently, $W = \mathcal{N}(M,\beta)$. □

We like to conclude this paper by comparing the nonweight irreducible Virasoro modules $\mathcal{N}(M,\beta)$ with other known irreducible Virasoro modules. When $\dim M = 1$, $\mathcal{N}(M,\beta)$ are exactly the modules studied in [LGZ]. The following result handles the remaining cases.

Comp **Theorem 9.** *Let $r \geq 1$, $\beta \in \mathbb{C}[t,t^{-1}] \setminus \mathbb{C}$, and M be a simple module over $\mathfrak{a}_r$ such that the action of $\bar{d}_r$ on M is injective. Then $\mathcal{N}(M,\beta)$ is a new irreducible Virasoro module.*

Proof. All other known non-weight Virasoro modules are from [LZ, MZ3, MW]. Let us recall those modules defined in [MZ3]. Let $\mathfrak{V}_+ = \operatorname{span}\{d_i \,|\, i \in \mathbb{Z}_+\}$. Given $N \in \mathfrak{V}_+$-mod and $\theta \in \mathbb{C}$, consider the corresponding induced module $\operatorname{Ind}(N) := U(\mathfrak{V}) \otimes_{U(\mathfrak{V}_+)} N$ and denote by $\operatorname{Ind}_\theta(N)$ the module $\operatorname{Ind}(N)/(\mathtt{c} - \theta)\operatorname{Ind}(N)$. It was proved that $V = \operatorname{Ind}_\theta(N)$ is irreducible over $\mathfrak{V}$ if $d_k N = 0$ for all sufficiently large k. For any $v \in V$ we have $d_k v = 0$ for all sufficiently large k. Now

we use the operator $\omega_{l,m}^{(2r+2)}$ defined in (3.1). From (6.7) we know that $\omega_{l,m}^{(2r+2)} \circ w \neq 0$ for any nonzero $w \in \mathcal{N}(M,\beta)$ and any $l, m \in \mathbb{Z}$, while for any nonzero $v \in V$ we have $\omega_{l,m}^{(2r+2)} v = 0$ if $l > 2m >> 0$. Thus $\mathcal{N}(M,\beta)$ is not isomorphic to V.

Now we compare our module $\mathcal{N}(M,\beta)$ with the irreducible Virasoro modules W $=\mathrm{Ind}_{\theta,z}(\mathbb{C}_{\mathbf{m}})$ defined in [MW], where $\theta, m_2, m_3, m_4 \in \mathbb{C}$ and $z \in \mathbb{C}^*$ satisfying the conditions

$$zm_3 \neq m_4, 2zm_2 \neq m_3, 3zm_3 \neq 2m_4, z^2 m_2 + m_4 \neq 2zm_3. \tag{6.8}$$

There is a nonzero vector $v \in W$ such that

$$(d_2 - zd_1)v = m_2 v, \ \ (d_3 - z^2 d_1)v = m_3 v, \ \ (d_4 - z^3 d_1)v = m_3 v.$$

And

$$(d_i - z^{i-1} d_1)v = -(i-4)m_3 z^{i-3} v + (i-3)m_4 z^{i-4} v, \ \forall\ i > 4.$$

Then

$$\begin{aligned} &\big(d_{i+1} - 2zd_i + z^2 d_{i-1}\big)\, v \\ =& -(i-3)m_3 z^{i-2} v + (i-2)m_4 z^{i-3} v \\ &+ 2(i-4)m_3 z^{i-2} v - 2(i-3)m_4 z^{i-3} v \\ &- (i-5)m_3 z^{i-2} v + (i-4)m_4 z^{i-3} v \\ =& \ 0, \ \forall\ i \geq 4. \end{aligned}$$

By (3.1), we see that

$$\begin{aligned} &(\omega_{5,5}^{(2r+2)} - 2z\omega_{4,4}^{(2r+2)} + z^2 \omega_{3,3}^{(2r+2)})v \\ =& \sum_{i=0}^{2r+2} \binom{2r+2}{i} (-1)^{2r+2-i} d_{-i}(d_{5+i} - 2zd_{4+i} + z^2 d_{3+i})v = 0. \end{aligned}$$

While we know from (6.7) that $(\omega_{5,5}^{(2r+2)} - 2z\omega_{4,4}^{(2r+2)} + z^2 \omega_{3,3}^{(2r+2)}) \circ w \neq 0$ for any nonzero $w \in \mathcal{N}(M,\beta)$. So $\mathcal{N}(M,\beta)$ is not isomorphic to W.

At last we compare our module $\mathcal{N}(M,\beta)$ with the irreducible Virasoro modules A_b defined in [LZ] where $b \in \mathbb{C} \setminus \{1\}$ and A_b is an irreducible module over the associative algebra $\mathbb{C}[t, t^{-1}, \frac{d}{dt}]$ such that $\mathbb{C}[\frac{d}{dt}]$ is torsion-free on A_b (otherwise A_b is a weight module). Recall that $\partial = t\frac{d}{dt}$. From (3.5) in [LZ] we deduce that, for any $v \in A_b$,

$$\begin{aligned} \omega_{l,m}^{(0)} \circ v &= d_{l-m} \circ d_m \circ v \\ &= \big(t^{l-m}\partial + (l-m)bt^{l-m}\big)(t^m \partial + mbt^m)v \\ &= t^l(\partial^2 + (m+lb)\partial + (m(1-b)+lb)mb)v. \end{aligned}$$

Then we can compute

$$\omega_{l,m}^{(s)} \circ v = \sum_{i=0}^{s} \binom{s}{i} (-1)^{s-i} d_{l-m-i} \circ d_{m+i} \circ v$$
$$= t^l \sum_{i=0}^{s} \binom{s}{i} (-1)^{s-i} \left(\partial^2 + (m+i+lb)\partial + ((m+i)(1-b)+lb)(m+i)b\right) v$$
$$= t^l \sum_{i=0}^{s} \binom{s}{i} (-1)^{s-i} \left(i\partial + i^2(1-b)b + b(2m(1-b)+lb)i\right) v$$
$$= 0, \ \forall\, l,m \in \mathbb{Z}, s \geq 3, v \in A_b.$$

From (6.7) we know that our module $\mathcal{N}(M,\beta)$ does not satisfy this property. Thus $\mathcal{N}(M,\beta)$ is not isomorphic to any A_b. This completes the proof. □

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